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Modeling cooperation on a class of distribution problems

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ABSTRACT

In this paper we study models of cooperation between the nodes of a network that represents a distribution problem. The distribution problem we propose arises when, over a graph, a group of nodes offers certain commodity, some other nodes require it and a third group of nodes neither need this material nor offer it but they are strategically relevant to the distribution plan. The delivery of one unit of material to a demand node generates a fixed profit, and the shipping of the material through the arcs has an associated cost. We show that in such a framework cooperation is beneficial for the different parties. We prove that the cooperative situation arising from this distribution problem is totally balanced by finding a set of stable allocations (in the core of an associated cooperative game). In order to overcome certain fairness problems of these solutions, we introduce two new solution concepts and study their properties.

1. Introduction

Operations research has studied distribution models from its early years. The goal has been to find solutions optimizing a given objective function related to the cost or benefit induced by the model. In this direction, optimization problems over graphs are extensively used in real applications to model situations like production planning, communication, scheduling, transportation or assignment among others. In such problems, it is normally assumed that the resources used in the model are under the control of a single decision maker or a group of decision makers having identical interests.

However, we often find situations in which the resources are owned by several agents with conflicting objectives, which may consider cooperating with each other in order to obtain a better global solution. Situations like that arise, for instance, when there is a group of warehouses having a certain commodity and several retailers where that commodity can produce benefits. The transportation of the material from one point to another generates costs and the warehouses together with the retailers have to decide how to distribute the commodity in order to obtain the highest profit. It directly follows that cooperative game theory can be used to analyze such situations and to find fair allocations of the collective profit that the agents can make.

In this paper we study a model of cooperation over such a supply chain problem. Supply chain problems are among the most complex models analyzed nowadays by operations research, see Borm et al. [4]. They include different aspects ranging from location or distribution to scheduling or inventory management. The optimization phase of supply chain problems has been widely studied from most of its different perspectives (the interested reader is referred to Brewer et al. [3]). Nevertheless, cooperation in supply chain models has been addressed mainly from an inventory point of view, see the recent survey by Meca and Timmer [20]. These models assume that firms can make binding agreements on the convenience of the entire system and studies consolidation in different costs of the system. The interested reader is referred to Eppen [5], Gerchak and Gupta [8], Hartman and Dror [11–13], Hartman et al. [14], Anupindi et al. [1], Müller et al. [22], Meca et al. [19,18], Minner [21], Ozen et al. [25], Tijs et al. [33] and Slikker et al. [32] among others, for comprehensive literature on this subject.

In this work we address a different aspect of the model and we focus on distribution matters rather than in inventory control. Several models of cooperation on graphs have been studied in the literature, see Fragnelli et al. [6], Markakis and Amin [17], Myerson [23], Puerto et al. [27], Sánchez-Soriano et al. [29] or Voorneveld and Grahn [34], among others. In this respect, our problem could be cast as a network flow problem, thus it is related with the literature in this field, see for instance Granot and Granot [9] and Kalai and Zemel [16].

Comparing with the above approaches, the model in this paper incorporates the fact that suppliers, intermediary centres and retailers are players, and that the purchasing costs, production costs and selling benefits are player dependant. We prove that cooperation is advantageous for the firms in the chain. Moreover, we find that this type of cooperation is stable, i.e. there are fair divisions of the overall benefit of the distribution system among the agents such that no group of them would like to leave the chain. A previous approach in this direction can be found in

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Guardiola et al. [10], where the problem in which a single supplier supplies several retailers with goods for replenishment of stocks is studied via cooperative game theory.

Needless to say, none of the previously mentioned approaches by themselves are enough to perform a complete analysis of a supply chain as a whole. In this regard, our approach can be seen as a new building block that, together with all the previous attempts, will help in understanding the complex nature of cooperation in supply chain models.

The paper is structured as follows. In Section 2 we present our model of distribution problem (*DiP*) and its formulation as a linear program. Section 3 is devoted to introduce the class of cooperative games arising from *DiP*, called distribution games (*DiG*), and some of its properties are shown. We prove that the core is always nonempty by proposing a core allocation for *DiG* that, additionally, can be computed in polynomial time. Due to the fact that such a core allocation may give null payoff to players that are absolutely necessary for any profit to be made, in Section 4 we introduce other allocation rules for our class of distribution games. The paper finishes with some conclusions and an outlook for further research.

2. A distribution problem

The standard transportation problem arises when an optimal distribution plan to transport a commodity on a bipartite network $G = (N, A)$, N being the set of nodes (the set of nodes N is divided into two disjoint groups P, Q) and A the set of arcs, must be determined. The nodes of P offer that commodity and the nodes of Q require the same commodity, and there is an arc joining each node that offers material with each node demanding it. In a transportation problem it is assumed that the shipping of one unit of material from a node in P to a node in Q gives rise to a profit, the goal being to maximize the total profit generated when covering the demand.

Assume now that we are given a directed network $G = (N, A)$, where N and A are the set of nodes and the set of arcs of the graph, respectively. Assume further that each node $i \in N$ has two scalar numbers $b_i \in \mathbb{R}$ and $k_i \in \mathbb{R}_+$, associated with it. If b_i is positive, node i is said to be a *supply* node ($i \in P$), if it is negative we say that node i is a *demand* node ($i \in Q$) and if it is null we say that node i is a *transfer* node ($i \in R$). Consider the following sets:

$$P := \{i \in N : b_i > 0\}, \quad Q := \{i \in N : b_i < 0\}, \quad R := \{i \in N : b_i = 0\}.$$

We shall call these sets *supply* set, *demand* set and *transfer* set, respectively. It is clear that P, Q and R constitute a partition of N. A supply node i can produce up to b_i units of certain material at a unitary cost of k_i. A demand node j can sell up to $-b_j$ units of the material at a unitary price of k_j. Transfer nodes cannot produce nor sell material, but they can be used as intermediate points from supply nodes to demand nodes in the distribution plan. Set $k_i = 0$ for every transfer node.

We denote each arc of A by the ordered pair consisting of its initial and final nodes, that is, the arc (i, j) joins nodes i and j in this way. Each arc (i, j) has a scalar number $c_{ij} \in \mathbb{R}$ associated with it, which is interpreted as the necessary cost of the shipping of one unit of material through the arc (i, j). When shipping directly from i to j is not possible we assume that the cost of the arc c_{ij} is $+\infty$. Besides, the capacity of arc (i, j) is bounded from above by $h_{ij} \in [0, +\infty]$ $\forall (i, j) \in A$. The problem consists of finding a feasible distribution plan that maximizes the total benefit. Thus, given the set of nodes N, where $|N| = n$, the set of arcs $A \subset N \times N$, two matrices $C, H \in \mathbb{R}^{n \times n}$, and two vectors $b, k \in \mathbb{R}^n$, we denote our distribution problem (*DiP*) as the 6-tuple (N, A, C, b, k, H).

As an example of this situation, consider a railway network in which locomotives must be sent from the factories where they are made (at a certain cost) to the stations where they will be used (generating a certain benefit). In order to send the locomotives from the factories to the stations, the railway system must be used, and therefore costs (related to the necessary fuel or other energy sources) must be met. The problem therefore consists of deciding how to send the locomotives from the factories to the stations, so that the total benefit of the system is maximized.

Example 2.1. Consider the *DiP* as depicted in Fig. 1. It consists of 3 nodes, therefore $N = \{1, 2, 3\}$. We have that node 1 is a supply node, node 2 is a transfer node and node 3 is a demand node, thus $P = \{1\}, Q = \{3\}, R = \{2\}$. There is an arc joining nodes 1 and 2, whose associated cost equals 1, and another arc joining nodes 2 and 3, whose associated cost equals 1 as well (the costs of the arcs are indicated by the numbers over them). Besides, none of the arcs has capacity constraints. That is, $A = \{(1, 2), (2, 3)\}$ and $c_{12} = c_{23} = 1, h_{12} = h_{23} = +\infty$. Notice that when $h_{ij} = +\infty$ we do not represent it. Node 1 offers 1 unit of material, which can be produced at a cost of 1 monetary unit, and node 3 would accept up to 2 units, which would produce a benefit of 7 monetary units each, therefore $b = (1, 0, -2)$ and $k = (1, 0, 7)$ (b_i and k_i are represented by the 2-component vectors by the nodes in $P \cup Q$, nodes in R do not have demand nor offer).

Note that, unlike the transportation problem, the network that gives rise to a *DiP* does not have to be bipartite. Besides, we include in our model of *DiP* a new kind of nodes, transfer nodes, which do not offer material nor require it.

In addition, we remark that in the proposed distribution model it is not necessary to cover all the demand nor to launch all the available offer. In a general *DiP*, the demand of a node j will be covered if and only if there exists a profitable path from the supply nodes to j and there is some available material. Situations like this may not arise when the goal of the network is to satisfy basic needs such as medical care, educational centers, etc. Thus, our model is not appropriate to cover "social" networks, since the active agents in *DiP* are "selfish" in the sense that their only goal is to maximize their own benefit. It is not difficult to find examples of *DiP* in which optimal distribution plans do not cover all the demand or do not launch all the material that supply nodes can produce, or both cases at the same time, as for instance in Example 2.1, where node 3 will never cover all its demand.

In the rest of the section we present the linear programming (LP for short) formulation of a general *DiP*. Let x_{ij} be the amount of shipped-through-arc-(i, j) material, $\forall (i, j) \in A$. A feasible distribution plan $x = (x_{ij})_{(i,j) \in A}$ must satisfy several conditions:

- Supply nodes cannot produce new material; that is, the amount of material leaving from a certain supply node, *outgoing* material, minus the amount of material that goes to it, *incoming* material, must be less than or equal to what the node can offer. Besides, supply nodes should not keep new material. Thus we have that

$$0 \leqslant \sum_{j \in N : (i,j) \in A} x_{ij} - \sum_{k \in N : (k,i) \in A} x_{ki} \leqslant b_i \quad \forall i \in P. \tag{1}$$

- In our model, demand nodes cannot receive more than what they request, since they cannot sell more units than required. Thus, the amount of incoming flow minus the amount of outgoing flow must be less than or equal to its demand. Further, demand nodes do not have the capability to create new material

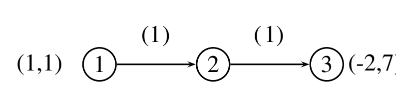

Fig. 1. The transportation network of Example 2.1.

$$0 \leqslant \sum_{k\in N:(k,i)\in A} x_{ki} - \sum_{j\in N:(i,j)\in A} x_{ij} \leqslant -b_i \quad \forall i \in Q. \tag{2}$$

- The incoming material must be equal to the outgoing material in every transfer node, that is, transfer nodes can neither create nor keep material

$$\sum_{j\in N:(i,j)\in A} x_{ij} - \sum_{k\in N:(k,i)\in A} x_{ki} = 0 \quad \forall i \in R. \tag{3}$$

- Besides, the flow must be non-negative and satisfy the capacity constraints

$$0 \leqslant x_{ij} \leqslant h_{ij} \quad \forall (i,j) \in A. \tag{4}$$

As for the objective function, three main components must be considered:

- We want to maximize the total benefit, that is, the total demand covered in each node times the unitary profit that can be made there

$$\sum_{i\in Q} k_i \left(\sum_{k\in N:(k,i)\in A} x_{ki} - \sum_{j\in N:(i,j)\in A} x_{ij} \right). \tag{5}$$

- The costs of producing material are to be minimized

$$\sum_{i\in P} k_i \left(\sum_{j\in N:(i,j)\in A} x_{ij} - \sum_{k\in N:(k,i)\in A} x_{ki} \right). \tag{6}$$

- The transportation through each arc gives rise to a cost, so we must minimize

$$\sum_{(i,j)\in A} c_{ij} x_{ij}. \tag{7}$$

Joining Eqs. (5)–(7) the objective function is to maximize

$$\sum_{(i,j)\in A} (k_j - k_i - c_{ij}) x_{ij}. \tag{8}$$

To summarize, given a *DiP* (N,A,C,b,k,H), where $N = \{1,\ldots,n\}$, $A \subset N \times N, C, H \in \mathbb{R}^{n\times n}$, and $b, k \in \mathbb{R}^n$, an optimal distribution plan is given by any optimal solution to the linear program consisting of maximizing (8) under the constraints (1)–(4).

Example 2.2. Consider the *DiP* introduced in Example 2.1. Its formulation as a linear problem is

$$\begin{aligned}
\max \quad & -2x_{12} + 6x_{23} \\
\text{s.t.} \quad & 0 \leqslant x_{12} \leqslant 1 \\
& 0 \leqslant x_{23} \leqslant 2 \\
& x_{12} - x_{23} = 0 \\
& x_{12}, x_{23} \geqslant 0,
\end{aligned}$$

yielding a unique optimal distribution plan $x^*_{12} = x^*_{23} = 1$, which generates an optimal benefit of 4 units.

Note that any *DiP* can be reformulated as a Network Flow Problem, by adding two new fictitious nodes, s and t, an arc from s to each supply node with capacity equal to the offer of that supply node, and an arc from each demand node to t with a capacity equal to the required demand of the corresponding demand node. However, we will use the previous formulation since its structure will be more adequate to analyze the cooperation arising from *DiP*, as introduced in the following section. The reason is because with this formulation we can identify each player with a unique node of the network whereas with the flow formulation players must be identified with subsets of arcs.

3. Distribution games

After having defined our class of distribution problems, we are now interested in studying the model that arises when each of the nodes in the distribution network belongs to a different agent, see Perea [26] for a review of graph games. In this situation, it makes sense to wonder whether cooperation among the agents, sharing products and using the network, leads to a stable distribution model or whether subgroups of agents will split into smaller distribution systems. This analysis is performed by means of cooperative game theory.

A cooperative game is a pair (N, v) where $N = \{1,\ldots,n\}$ is the set of players and v is the characteristic function: $v : 2^N \to \mathbb{R}$, and $v(\emptyset) = 0$; where for every $S \subseteq N$, $v(S)$ can be seen as the maximum profit that the coalition S can make by acting on its own. So, $v(N)$ is the best payoff that the coalition of all players can make. This coalition, N, is called *the grand coalition.*

Therefore, given a *DiP* (N,A,C,b,k,H) we define the corresponding distribution game (*DiG*) as follows. The set of players is $N = \{1,\ldots,n\}$, every node is owned by one player and each player is associated with the only node that it owns. Then, we denote the player that owns node i by i, $\forall i \in N$. Thus, we have supply players (set P), demand players (set Q) and transfer players (set R).

As for the characteristic function v of the game, each coalition $S \subset N$ generates a new *DiP* $(S, A_S, C_S, b_S, k_S, H_S)$, where $A_S = (S \times S) \cap A$, and C_S, H_S, b_S and k_S are the restrictions of C, H, b and k to S, respectively. In the same way, we have the supply, demand and transfer sets of S, defined as: $P_S = P \cap S, Q_S = Q \cap S, R_S = R \cap S$. The above subnetwork has an optimal distribution plan that yields the maximum profit that the coalition S can make. Hence, $v(S)$ is the optimal value of the following linear program

$$\begin{aligned}
\max \quad & \sum_{(i,j)\in A_S} (k_j - k_i - c_{ij}) x_{ij} \\
\text{s.t.} \quad & 0 \leqslant \sum_{j:(i,j)\in A_S} x_{ij} - \sum_{k:(k,i)\in A_S} x_{ki} \leqslant b_i \quad \forall i \in P_S, \\
& 0 \leqslant \sum_{k:(k,i)\in A_S} x_{ki} - \sum_{j:(i,j)\in A_S} x_{ij} \leqslant -b_i \quad \forall i \in Q_S \\
& \sum_{j:(i,j)\in A_S} x_{ij} - \sum_{k:(k,i)\in A_S} x_{ki} = 0 \quad \forall i \in R_S, \\
& 0 \leqslant x_{ij} \leqslant h_{ij} \quad \forall (i,j) \in A_S.
\end{aligned} \tag{9}$$

This problem is denoted by $LP(S)$ $\forall S \subset N$.

Definition 3.1. Let (N,A,C,b,k,H) be a *DiP*. The associated distribution game (*DiG*) is the cooperative game (N, v), where $v(S)$ is the optimal value of problem $LP(S)$ for every nonempty coalition $S \subseteq N$, and $v(\emptyset) = 0$.

It is easy to check that *DiG* are well defined and non-negative. To prove so just take $x_{ij} = 0$ for any arc $(i,j) \in A_S$, which is a feasible solution to problem $LP(S)$ for every coalition S.

Apart from non-negativity, there are some other interesting properties of *DiG* that ensure that players collaborate with each other:

(1) [0-normality] (N, v) is 0-normalized if $v(\{i\}) = 0 \quad \forall i \in N$.
(2) [Monotonicity] (N, v) is monotonic if $\forall S \subset T \subset N, \quad v(S) \leqslant v(T)$.
(3) [Superadditivity] (N, v) is superadditive if $\forall S, T \subset N : S \cap T = \emptyset \Rightarrow v(S) + v(T) \leqslant v(S \cup T)$.

Proposition 3.1. *Every DiG is 0-normalized, superadditive and monotonic.*

Those three properties confirm that cooperating is profitable in a *DiG*, as the benefit that players obtain when they join their resources increases. To prove that *DiG* are 0-normalized, just note that $A_{\{i\}} = 0$ for every $i \in N$. The proof of the other two properties follows from Theorem 3.1.

The reader can easily check that transportation games, as introduced by Sánchez-Soriano in [28,29], are a subclass of our class of distribution games. The contrary is not true, as we can see in the following example.

Example 3.1. Consider the *DiG* (N, v) arising from the *DiP* introduced in Example 2.1. One can see that $v(1,2,3) = 4$, $v(S) = 0$ for any other coalition. It is easy to see that, for any transportation game satisfying that $v(N) > 0$, there must be a pair of players i,j such that $v(i,j) > 0$, which does not hold in this example. Therefore, this example shows that *DiG* cannot always be cast as transportation games.

Since assignment games (*AG*), see Shapley and Shubik [31], are transportation games, it also follows that $AG \subseteq DiG$.

As a consequence of Proposition 3.1, we can also state that the class of shortest path games (*SPG*) includes the class of *DiG*. This is easily proven due to the fact that *SPG* coincide with the class of monotonic games, see [6]. The converse is not true, since the class of *SPG* is not totally balanced and the class of *DiG* will be proven to be totally balanced, see Corollary 3.1. All the above positions the class of distribution games within other well-known classes of games: $AG \subset TG \subset DiG \subset SPG$.

Among the most important problems we face when dealing with cooperative games where the grand coalition is to form is how to divide the total benefit among the players, that is, how to allocate $v(N)$. We define an allocation for the game (N, v) as a vector $\alpha \in \mathbb{R}^n$, where its ith coordinate represents the payoff that player i receives from the allocation α. An acceptable property of allocations is the *collective rationality principle*, which ensures that every coalition S of N receives from an allocation at least what it would obtain by acting without the help of other players in $N \setminus S$. Allocations satisfying that principle are called *core* allocations. The core of the game (N, v), denoted $C(N, v)$, is the set $\{\alpha \in \mathbb{R}^n : \alpha(S) \geqslant v(S)\ \forall S \subset N, \alpha(N) = v(N)\}$, where $\alpha(S) = \sum_{i\in S}\alpha_i$, that is, the payoff that coalition S receives from allocation α. We remark that $\alpha \in C(N, v)$ if and only if no coalition can improve upon α. Thus, each member of the core consists of a highly stable payoff distribution. For a cooperative game, having non-empty core is equivalent to being balanced, see Shapley [30]. Unfortunately not all cooperative games have core allocations.

In our analysis of *DiG* the following step is to prove that these games have non-empty core. This property will ensure that cooperation is stable and no agent will have incentives to leave coalitions. To this end, we provide a procedure to find a core allocation for every *DiG*.

In general, there is no polynomial time algorithm for finding allocations in the core of a cooperative game (N, v) since we have to solve a LP problem of the form $\{\min \sum_{i=1}^n x_i | \sum_{i\in S} x_i \geqslant v(S)\ \forall S \subset N\}$, which is a linear programming problem with n variables and $O(2^n)$ constraints. That is why finding core allocations in an efficient way is a crucial issue.

In the following we give a procedure to find a core allocation for *DiG* in polynomial time. The characteristic function of a *DiG* is given by the optimal value of the linear problem $LP(S)$. One can check that its dual problem (see Bazaraa et al. [2]), called from now on $D(S) \forall S \subset N$, is the following linear program:

$$\begin{array}{ll}
\min & \sum_{i\in P_S} b_i u_i - \sum_{j\in Q_S} b_j u_j + \sum_{(i,j)\in A_S^*} v_{ij} h_{ij} \\
\text{s.t.} & (u_i - t_i) - (u_j - t_j) + v_{ij} \geqslant k_j - k_i - c_{ij} \quad \forall\{i,j\} \in P_S \times P_S \\
& (u_i - t_i) + (u_j - t_j) + v_{ij} \geqslant k_j - k_i - c_{ij} \quad \forall\{i,j\} \in P_S \times Q_S \\
& (u_i - t_i) - u_j + v_{ij} \geqslant k_j - k_i - c_{ij} \quad \forall\{i,j\} \in P_S \times R_S \\
& -(u_i - t_i) - (u_j - t_j) + v_{ij} \geqslant k_j - k_i - c_{ij} \quad \forall\{i,j\} \in Q_S \times P_S \\
& -(u_i - t_i) + (u_j - t_j) + v_{ij} \geqslant k_j - k_i - c_{ij} \quad \forall\{i,j\} \in Q_S \times Q_S \\
& -(u_i - t_i) - u_j + v_{ij} \geqslant k_j - k_i - c_{ij} \quad \forall\{i,j\} \in Q_S \times R_S \\
& u_i - (u_j - t_j) + v_{ij} \geqslant k_j - k_i - c_{ij} \quad \forall\{i,j\} \in R_S \times P_S \\
& u_i + (u_j - t_j) + v_{ij} \geqslant k_j - k_i - c_{ij} \quad \forall\{i,j\} \in R_S \times Q_S \\
& u_i - u_j + v_{ij} \geqslant k_j - k_i - c_{ij} \quad \forall\{i,j\} \in R_S \times R_S \\
& u_i, t_i \geqslant 0 \quad \forall i \in P_S \cup Q_S \\
& v_{ij} \geqslant 0 \quad \forall (i,j) \in A_S^*
\end{array} \tag{10}$$

where $A_S^* = \{(i,j) \in A_S : h_{ij} < +\infty\}\ \forall S \subset N$. Note that variables v_{ij} do not make sense when $h_{ij} = +\infty$ and that the above constraints are only valid for those pairs $\{i,j\}$ such that $(i,j) \in A_S$. Note as well that $D(S)$ is always feasible and bounded, since $LP(S)$ has optimal feasible solutions. (An illustrative instance of this linear program can be found in Example 3.2.)

Theorem 3.1. *Let $((u^*)_i, (t^*)_i, (v^*)_{ij})$ be an optimal feasible solution to Problem (10) for $S = N$. Then, the allocation α, where*

$$\alpha_i = |b_i| u_i^* + \frac{1}{2}\left\{\sum_{j:(i,j)\in A_N^*} h_{ij} v_{ij}^* + \sum_{j:(j,i)\in A_N^*} h_{ji} v_{ji}^*\right\} \quad \forall i \in N, \tag{11}$$

is a core allocation.

Proof

1. First we see that such an allocation is efficient

$$\begin{aligned}
\alpha(N) &= \sum_{i\in N}\alpha_i = \sum_{i\in N}\left(|b_i|u_i^* + \frac{1}{2}\left\{\sum_{j:(i,j)\in A_N^*} h_{ij} v_{ij}^* + \sum_{j:(j,i)\in A_N^*} h_{ji} v_{ji}^*\right\}\right) \\
&= \sum_{i\in P} b_i u_i^* + \sum_{j\in Q}(-b_j)u_j^* + \frac{1}{2}\sum_{i\in N}\sum_{j:(i,j)\in A_N^*} h_{ij} v_{ij}^* + \frac{1}{2}\sum_{i\in N}\sum_{j:(j,i)\in A_N^*} h_{ji} v_{ji}^* \\
&= \sum_{i\in P} b_i u_i^* - \sum_{j\in Q} b_j u_j^* + \frac{1}{2}\sum_{(i,j)\in A_N^*} h_{ij} v_{ij}^* + \frac{1}{2}\sum_{(j,i)\in A_N^*} h_{ji} v_{ji}^* \\
&= \sum_{i\in P} b_i u_i^* - \sum_{j\in Q} b_j u_j^* + \sum_{(i,j)\in A_N^*} h_{ij} v_{ij}^* = v(N).
\end{aligned} \tag{12}$$

2. Now it will be proven that no coalition can improve the payoff they receive from α by acting by themselves

$$\begin{aligned}
\alpha(S) &= \sum_{i\in S}\alpha_i \\
&= \sum_{i\in S}|b_i|u_i^* + \frac{1}{2}\sum_{i\in S}\sum_{j:(i,j)\in A_N^*} h_{ij} v_{ij}^* + \frac{1}{2}\sum_{i\in S}\sum_{j:(j,i)\in A_N^*} h_{ji} v_{ji}^*.
\end{aligned} \tag{13}$$

Given $i \in S$, one has that $\{j : (i,j) \in A_S^*\} \subset \{j : (i,j) \in A_N^*\}$ and $\{j : (j,i) \in A_S^*\} \subset \{j : (j,i) \in A_N^*\}$. It is also clear that $h_{ij} v_{ij}^* \geqslant 0\ \forall (i,j) \in A_N^* \supseteq A_S^*$. Thus, we have that (13) is greater than or equal to

$$\begin{aligned}
&\sum_{i\in S}|b_i|u_i^* + \frac{1}{2}\sum_{i\in S}\sum_{j:(i,j)\in A_S^*} h_{ij} v_{ij}^* + \frac{1}{2}\sum_{i\in S}\sum_{j:(j,i)\in A_S^*} h_{ji} v_{ji}^* \\
&= \sum_{i\in P_S} b_i u_i^* - \sum_{i\in Q_S} b_i u_i^* + \frac{1}{2}\sum_{(i,j)\in A_S^*} h_{ij} v_{ij}^* + \frac{1}{2}\sum_{(j,i)\in A_S^*} h_{ji} v_{ji}^* \\
&= \sum_{i\in P_S} b_i u_i^* - \sum_{i\in Q_S} b_i u_i^* + \sum_{(i,j)\in A_S^*} h_{ij} v_{ij}^*.
\end{aligned} \tag{14}$$

Let (u^{*S}, t^{*S}, v^{*S}) be an optimal solution to $D(S)$. We obviously have that (u^*, t^*, v^*) is feasible for $D(S)$. Then we deduce that (14) is greater than or equal to

$$\sum_{i\in P_S} b_i u_i^{*S} - \sum_{i\in Q_S} b_i u_i^{*S} + \sum_{(i,j)\in A_S^*} h_{ij} v_{ij}^{*S} = v(S). \tag{15}$$

Thus, we have proven that

$$\alpha(S) \geqslant v(S) \quad \forall S \subset N. \tag{16}$$

By joining (1 and 2) we conclude that α is a core allocation of the game. □

An interpretation of this allocation is that α assigns a payoff to players that depends on the amount of offer or demand that they have, and on the amount of material running through their arcs, provided that those arcs have finite capacity.

Since every subgame of a *DiG* is also a *DiG*, the following result follows.

Corollary 3.1. *DiG are totally balanced.*

This fact assures that *DiG* are superadditive (since total balancedness implies superadditivity), and besides it implies that *DiG* are a subclass of flow games, see Kalai and Zemel [15], and linear production games, see Owen [24] (since flow games and linear production games coincide with the class of totally balanced games).

The contrary is not true, that is, there are *LPG* and *FG* that are not *DiG*. Take a *LPG* game (N, v) such that $C(N, v) \neq \emptyset$ and $v(\{i\}) > 0$ for some $i \in N$. On the one hand, this game is a *FG* since it is totally balanced. On the other hand, it cannot be a *DiG*, since *DiG* have been proven 0-normalized, see Proposition 3.1.

The set consisting of all the allocations that can be obtained from optimal solutions to $D(N)$ following the process given above is somehow similar to the Owen Set, defined for the class of linear production games in [24].

One could be tempted to think that all core allocations in *DiG* can be obtained from solutions to the dual problem. The following example shows that this is not true in general. In other words, the set of allocations *'a la' Owen*, i.e. obtained by dual optimal solutions, is a "proper" subset of $C(N, v)$ within the class of *DiG*.

Example 3.2. Take the *DiG* (N, v) in Example 2.1. It is easy to check that any allocation $x \in \mathbb{R}^3$ such that $x_1 + x_2 + x_3 = 4$, with non-negative entries, is a core allocation. On the other hand, in this case $D(N)$ is

$$\begin{aligned}
\min \quad & u_1 + 2u_3 \\
\text{s.t.} \quad & (u_1 - t_1) - u_2 \geqslant -2 \\
& u_2 + (u_3 - t_3) \geqslant 6 \\
& u_i, t_i \geqslant 0 \quad i = 1, 2, 3.
\end{aligned}$$

The unique optimal solution to that problem is $u_1^* = 4, u_2^* = 6$ the other variables being null. Therefore, for this example, the unique allocation obtained via dual solutions is $\{(4,0,0)\}$, which concludes that this set is strictly included in the core of the game.

4. Other allocations in *DiG*

Owen's type allocations are useful for showing the non-emptiness of the core of *DiG*. This solution set is characterized for LP games by van Gellekom et al. in [7]. Despite its axiomatic characterization and its computational efficiency, Owen's allocations should not be taken as ideal solutions. In *DiG*, one of the problems that such allocations present is that transfer nodes may receive nothing, even though they can be absolutely necessary to make any positive profit. Besides, from the complementary slackness theorem, supply players whose offer is not entirely used or demand players whose requirements are not completely fulfilled, may receive null payoff. As an example of this situation note that in Example 3.2 the allocation $\{(4,0,0)\}$, induces that players 2 and 3, which are absolutely necessary for any profit to arise, receive null payoff.

In order to overcome such problems we propose other solution concepts for our class of *DiG*.

4.1. The extended Owen allocations

In this section our goal is to overcome the disadvantages described in the above discussion. To this end, we associate with an optimal solution x^* to problem $LP(N)$ of any *DiP* its essential distribution situation, $DiP(x^*)$. The rationale behind this essential situation is that one cannot ensure the same benefit as the one obtained with $DiP(x^*)$ by using componentwise smaller resources (supply, demand and arc capacities) than those used in $DiP(x^*)$.

The following result states that to get an overall benefit equal to the one obtained by $LP(N)$, the minimum amount of resources that will be used cannot be componentwise smaller than:

$$\begin{aligned}
b_i^{x^*} &= \sum_{j:(i,j)\in A} x_{ij}^* - \sum_{j:(j,i)\in A} x_{ji}^* \quad \forall i \in P, \\
b_i^{x^*} &= \sum_{j:(j,i)\in A} x_{ji}^* - \sum_{j:(i,j)\in A} x_{ij}^* \quad \forall i \in Q, \\
h_{ij}^{x^*} &= x_{ij}^* \quad \forall (i,j) \in A,
\end{aligned} \tag{17}$$

which are the amount of material produced by nodes in P, the amount of material received by nodes in Q and the amount of material running through arcs in A, respectively, in the optimal distribution plan x^*. Note that $b_i^{x^*} = 0\ \forall i \in R$.

Proposition 4.1. *Let (N, A, C, b, k, H) be a DiP and x^* a non-degenerate optimal distribution plan. Then, it does not exist $((\bar{b}_i)_{i\in P}, (\bar{b}_i)_{i\in Q}, (\bar{h}_{ij})_{(i,j)\in A})$, satisfying that $(\bar{b}_i) \leqslant b_i^{x^*}\ \forall i \in P \cup Q$, $\bar{h}_{ij} \leqslant h_{ij}^*\ \forall (i,j) \in A$, with at least one strict inequality, such that the optimal value of $(N, A, C, \bar{b}, k, \overline{H})$ equals the optimal value of $(N, A, C, b^{x^*}, k, H^{x^*})$.*

Proof. Let us denote by B and g the technological matrix and the right-hand-side vector of problem $LP(N)$, respectively. Assume that $\bar{x} \neq x^*$ is another optimal solution. Thus, since x^* is non-degenerate, there must exist two constraints i, j in $LP(N)$ such that:

$$\begin{aligned}
s_i^* &= (Bx^* - g)_i = 0 \quad \text{and} \quad \bar{s}_i = (B\bar{x} - g)_i < 0 \\
s_j^* &= (Bx^* - g)_j < 0 \quad \text{and} \quad \bar{s}_j = (B\bar{x} - g)_j = 0.
\end{aligned}$$

Hence, the right-hand side vectors of the essential problems $DiP(\bar{x})$ and $DiP(x^*)$, $\bar{g} = g - \bar{s}$ and $g^* = g - s^*$, respectively, are non-comparable componentwise and therefore $((b_i^{\bar{x}})_{i\in P}, (b_i^{\bar{x}})_{i\in Q}, (h_{ij}^{\bar{x}})_{(i,j)\in A})$ cannot be componentwise smaller than $((b_i^{x^*})_{i\in P}, (b_i^{x^*})_{i\in Q}, (h_{ij}^*)_{(i,j)\in A})$. □

Next we define the essential game of a *DiG* associated to an optimal solution to the corresponding *DiP*.

Definition 4.1. Let (N, A, C, b, k, H) be a *DiP* and x^* an optimal distribution plan. Let (N, v) be the corresponding *DiG*. The essential game of (N, v) associated to x^*, denoted by (N, v^{x^*}), is the *DiG* arising from the *DiP* (N, A, C, b^*, k, H^*), where b^* and H^* are as defined in Eq. (17).

Note that in each of those games, players have their demand and offer reduced as much as possible while obtaining the same optimal benefit. Therefore, no surplus is generated in the optimal solution. The players also make agreements to reduce the capacity of the arcs. This way, there is no slack in the constraints of $LP(N)$ and the corresponding dual variables in $D(N)$ are not forced to be zero, thus the payoffs of the players related to those variables need not be zero either. Note as well that, as a consequence of Proposition 4.1, one cannot obtain the same optimal benefit as in $LP(N)$ with a componentwise smaller resource vector.

Moreover, the following result gives us a necessary condition for players to receive a strictly positive payoff in essential distribution games.

Theorem 4.1. *Let us assume that x^* is a non-degenerate optimal solution to DiP, its associated constraint is binding at x^* and let $f := (k_j - k_i - c_{ij})_{(i,j)\in A}$ denote the objective function vector. Player i will receive a positive payoff if the $|A| - 1$ constraints, different from agent's i constraint, that determine x^* and f are linearly independent.*

Proof. Assume that x^* is uniquely determined by the $|A|$ constraints indexed by the set I, and $i \in I$. Let $B(I)$ and $g(I)$ denote the matrix and right-hand-side vector of problem $LP(N)$ with rows in I, respectively. Thus, x^* is the unique solution to the system of equations $B(I)x = g(I)$.

Consider the following linear problem:

$$(P(I)) \quad \max \quad fx \qquad \text{s.t.} \quad B(I)x = g(I). \tag{18}$$

This problem has a unique feasible solution, namely x^*, and thus is finite. By the strong duality theorem in linear programming, its dual $D(I)$, is also finite, $D(I)$ being

$$(D(I)) \quad \min \quad u(I)g(I) \qquad \text{s.t.} \quad u(I)B(I) = f. \tag{19}$$

Let $u^*(I)$ be an optimal solution to $D(I)$. Let us see why $u(I)$ is made of non-null components. By contradiction, assume $u(I)_j = 0$ for some $j \in I$. Let $B(I \setminus j), u(I \setminus j)$ and $g(I \setminus j)$ be the corresponding elements, where the jth component is removed (row, variable or entry, respectively).

Note that problem $D(I \setminus j)$ has a unique feasible solution, $u^*(I \setminus j) = (u^*(I)_i)_{i\in I\setminus j}$, and the optimal value $u^*(I \setminus j)b(I \setminus j) = u^*(I)g(I) = fx^*$. Hence, its primal, $P(I \setminus j)$, must have a finite optimal solution. However, the feasible region is clearly an affine variety that contains a line passing through x^* and by hypothesis it is not included in $\{x : fx = 0\}$, since the constraints under consideration are linearly independent, which implies that $P(I \setminus j)$ is unbounded. This can be proven by considering that any feasible point is of the form $\bar{x} = x^* + \alpha t$, $\alpha \in \mathbb{R}$ and t being the director vector of the feasible region. Therefore, $f\bar{x} = fx^* + \alpha ft$, which can be made as large as we want.

The above contradiction proves that the dual solution cannot have null variables, and that all players in I will receive positive payoff by the allocation given by (11) built upon $u^*(I)$, in particular also i. □

Therefore, essential distribution games give us the possibility of assigning positive payoff to players that are useful for the optimal distribution payoff. For that reason, they will be used to define our next solution concept for distribution games, which is aimed at avoiding the problem of giving null payoff to players that are necessary for the maximum payoff (optimal distribution plan) to be made.

The extended Owen set, $EOwen(x^*)$, for the essential $DiG(x^*)$ is the set of allocations obtained by dual solutions from (N, v^{x^*}), i.e.

$$EOwen(x^*) = \{\alpha \in \mathbb{R}^n : \alpha_i \text{ as defined in } (11), \quad i = 1, \ldots, n\}.$$

It is easy to check that the name "extended Owen" is meaningful, that is, any allocation obtained *'a la' Owen* from (N, v) is included in $EOwen(x^*)$. Therefore, *EOwen* is always non-empty and $EOwen(x^*) \cap C(N, v) \neq \emptyset$. Besides, applying duality properties of linear programming one can easily prove that allocations in *EOwen* are efficient.

The following example illustrates the EOwen solution.

Example 4.1. Consider the distribution game of Example 2.1. The only optimal distribution plan is $x^*_{12} = x^*_{13} = 1$, therefore we obtain the essential situation as depicted in Fig. 2. Notice that now, in the essential game, the capacities of the arcs are 1 (the value of the flow in the optimal distribution plan, represented by the second component of the vectors over the arcs), and the offer of node 3 is reduced to 1.

The dual problem of the corresponding game (N, v^{x^*}) is

$$\begin{array}{ll} \min & u_1 + u_3 + v_{12} + v_{23} \\ \text{s.t.} & (u_1 - t_1) - u_2 + v_{12} \geqslant -2 \\ & u_2 + (u_3 - t_3) + v_{23} \geqslant 6 \\ & u_i, t_i \geqslant 0 \quad i = 1, 2, 3. \end{array}$$

This problem has 4 optimal extreme solutions, whose non-null variables are: $\{(u_1 = 4, u_2 = 6), (u_3 = 4, u_2 = 2), (v_{12} = 4, u_2 = 6), (v_{23} = 4, u_2 = 2)\}$, generating four allocations $\{(4, 0, 0), (0, 0, 4), (2, 2, 0), (0, 2, 2)\}$, respectively. The convex hull of those allocations constitute the extended Owen set. Note that each player can obtain a positive payoff from allocations in that set, unlike the allocation obtained from the dual program of the original game, $(4, 0, 0)$, where players 2 and 3 obtain null payoff.

In general, the relationship between the core, the set of allocations *'a la' Owen* and the extended Owen set in *DiG* can be summarized in Fig. 3.

4.2. Arc-proportional allocations

Another tailor-made allocation for our class of distribution games is the arc-proportional allocation. This form of sharing benefits in a *DiG* is an extension of the *Arc-Egalitarian* solution introduced for transportation games, see [28] (the reader may note that, as we previously mentioned, *DiG* are an extension of *TG*). In such an allocation, players are rewarded according to the amount of material running through their arcs in an optimal solution. That is, their payoffs depend on how active they are in an optimal distribution plan.

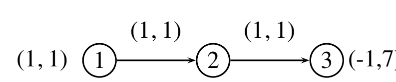

Fig. 2. The essential distribution problem.

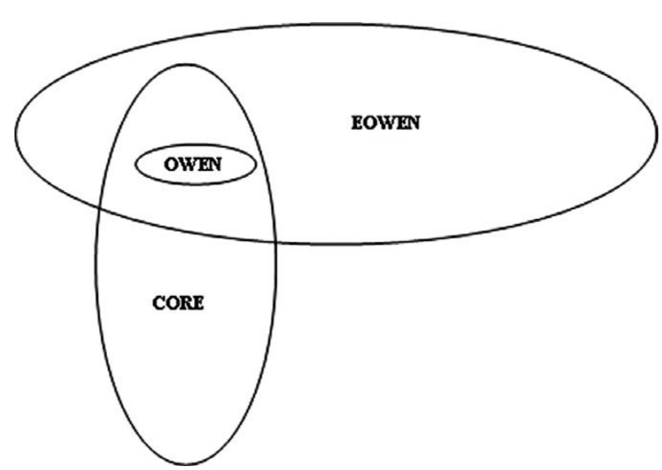

Fig. 3. The core, the EOwen set and the set of allocations *'a la' Owen*.

We start by giving its definition. Let (N,A,C,b,k,H) be a distribution problem, and let x^* be an optimal distribution plan. Let (N,v) be the associated game. The arc-proportional allocation $\gamma(x^*)$ of the game (N,v) is

$$\gamma_i(x^*) = \frac{1}{2}\sum_{j:(i,j)\in A}\left(\frac{L(x^*)}{T(x^*)} - c_{ij}\right)x^*_{ij} + \frac{1}{2}\sum_{j:(j,i)\in A}\left(\frac{L(x^*)}{T(x^*)} - c_{ji}\right)x^*_{ji}, \quad \forall i\in N, \tag{20}$$

where

$$L(x^*) = \sum_{(i,j)\in A}(k_j - k_i)x^*_{ij}, \quad T(x^*) = \sum_{(i,j)\in A}x^*_{ij}. \tag{21}$$

$L(x^*)$ and $T(x^*)$ are interpreted as the benefit without taking into account the transportation costs after the distribution plan x^* and the total amount of material transported between nodes, respectively. Note that both in $L(x^*)$ and $T(x^*)$ the material might be counted more than once, if it flows through different arcs from a supply node to a demand node. From now on, we may refer to γ, L and T instead of $\gamma(x^*), L(x^*)$ and $T(x^*)$ as long as it is clear that we refer to the optimal distribution plan x^*.

Example 4.2. In the *DiG* of Example 2.1, one has that $L(x^*) = 6$ and $T(x^*) = 2$. Therefore, one can see that the arc-proportional allocation is $\gamma = (1,2,1) \in C(N,v)$.

It can be easily proven that arc-proportional allocations are efficient. From its definition, it is also easy to prove that arc-proportional allocations satisfy the individual rationality principle if $\frac{L(x^*)}{T(x^*)} \geqslant c_{ij}\ \forall (i,j)\in A : x^*_{ij}\neq 0$. This condition can be interpreted as the material not "taking a long way" from the supply nodes to the demand nodes upon the optimal distribution plan. It also is easy to see that the arc-proportional solution satisfies the symmetry property, meaning that equal players receive equal payoffs from arc-proportional allocations. The last immediate consequence of the arc-proportional definition is that such allocations satisfy the standard property for two players, meaning that for every *DiG* $(\{1,2\},v)$ one has that $\gamma_i = \frac{v(\{1,2\})}{2},\ i = 1,2.$

Unfortunately, arc-proportional allocations are not in general core allocations. As an example take the *DiG* with the following data: $P = \{1,2\}, Q = \{3,4\}, R = \emptyset, A = \{(1,3),(1,4),(2,1),(2,3),(2,4),(3,4)\}, b = (4,3,-5,-2), k = (1,1,11,11)$, with arc-cost matrix

$$C = \begin{pmatrix} - & - & 1 & 5 \\ 4 & - & 6 & 7 \\ - & - & - & 4 \\ - & - & - & - \end{pmatrix},$$

and no capacity constraints on the arcs. One arc-proportional allocation for this game is $\gamma = (21.75, 4.125, 19.375, 1.75)$, thus $\gamma(\{2,4\}) = 5.875 < v(\{1,2\}) = 6$. Note that, since the previous example is a *TG*, the arc-proportional allocation and the Arc-Egalitarian allocation coincide.

The following proposition proves that, under some conditions, arc-proportional solutions are core allocations for distribution games. Condition 1 says that each supply node is directly connected by one arc to every demand node. Condition 2 states that the costs of sending one unit of material from a supply node to a demand node are constant. The third condition assures that shipping units via transfer nodes is not optimal. The forth condition says that the total amount of demand equals the total amount of available material. Condition 5 states that the costs of producing material and the benefits after receiving them are not dependent upon the nodes.

Proposition 4.2. *Let* (N,A,C,b,k,H) *be a distribution problem satisfying:*

1. $\{(i,j) : i\in P, j\in Q\} \subset A$.
2. $c_{ij} = c\ \forall i\in P, j\in Q$.
3. $c_{ij} > \frac{c}{2}\ \forall (i,j)\in A\setminus\{(i,j) : i\in P, j\in Q\}$.
4. $\sum_{i=1}^n b_i = 0$
5. $k_i = q_1\ \forall i\in P, k_j = q_2\ \forall j\in Q$,

and x^* *an optimal distribution plan of the corresponding DiP. The associated DiG* (N,v) *satisfies*

$$\gamma(x^*) \in C(N,v). \tag{22}$$

Proof. Under the conditions above, it is easy to see that supply nodes send all their material directly to demand nodes without passing through transfer nodes in any optimal distribution plan. Thus, γ (the arc-proportional solution associated to the optimal transportation x^*) is

$$\gamma_i = \frac{1}{2}\left\{\sum_{j\in N:(i,j)\in A}x^*_{ij}\left(\frac{L}{T} - c_{ij}\right) + \sum_{j\in N:(j,i)\in A}x^*_{ji}\left(\frac{L}{T} - c_{ji}\right)\right\} \quad \forall i\in N. \tag{23}$$

From the definitions of L and T, and the fifth condition of the theorem, we conclude that $\frac{L}{T} = q$, where $q = q_2 - q_1$. Joining this result with Eq. (23) we conclude that

$$\gamma_i = \frac{1}{2}(q-c)|b_i|\ \forall i\in N. \tag{24}$$

Now we shall see that γ is in the core of the game. The efficiency of the allocation is trivial. Let S be a coalition of N. On the one hand, since the members of S send as much as they can from P_S to Q_S, we have that

$$v(S) = (q-c)\min\left\{\sum_{i\in P_S}b_i, -\sum_{j\in Q_S}b_j\right\}. \tag{25}$$

On the other hand

$$\begin{aligned}
\gamma(S) &= \sum_{i\in S}\gamma_i = \sum_{i\in P_S}\gamma_i + \sum_{j\in Q_S}\gamma_j + \underbrace{\textstyle\sum_{k\in R_S}\gamma_k}_{=0} \\
&= \sum_{i\in P_S}\frac{1}{2}(q-c)b_i + \sum_{j\in Q_S} -\frac{1}{2}(q-c)b_j \\
&= \frac{1}{2}(q-c)\sum_{i\in P_S}b_i + \frac{1}{2}(q-c)\left(-\sum_{j\in Q_S}b_j\right) \\
&\geqslant \frac{1}{2}(q-c)\min\left\{\sum_{i\in P_S}b_i, -\sum_{j\in Q_S}b_j\right\} \\
&\quad + \frac{1}{2}(q-c)\min\left\{\sum_{i\in P_S}b_i, -\sum_{j\in Q_S}b_j\right\} \\
&= (q-c)\min\left\{\sum_{i\in P_S}b_i, -\sum_{j\in Q_S}b_j\right\} = v(S),
\end{aligned} \tag{26}$$

and that concludes the proof. □

In order to estimate the frequency of this solution being a core allocation, we performed a computational experiment. We built 800 random Distribution Games, with number of players varying from 3 to 10 (100 instances of each case). The existence of an arc joining nodes (i,j) follows a Bernouilly random variable of parameter 2/3. The parameters of the game were uniformly distributed in the following ranges: $c_{ij}\in[1,5], h_{ij}\in[6,10]\ \forall (i,j)\in A, b_i\in[-10,10]\ \forall i\in N, k_i\in[6,10],\ \forall i\in Q, k_i\in[1,5],\ \forall i\in P,$ all those parameters being integer.

The results showed that arc-proportional allocations are core allocations in almost 50% of the instances.

5. Concluding remarks

In this paper a new class of games arising from a particular distribution problem is presented. This type of cooperation is analyzed and the class of games is proved to be monotonic, totally balanced, and a polynomial time procedure to find allocations in its core is given. Since such allocations may give null payoff to transfer players, even though they might be veto players, other type of allocations overcoming those drawbacks are introduced.

We also established the relationship of *DiG* with other well-known classes of games: $DiG \subsetneq FG = LPG$, $AG \subsetneq TG \subsetneq DiG$ and $DiG \subsetneq SPG$. Hence, clearly establishing the position of this new class of games in relation with previously known cooperative games.

Further research on this topic is focusing on axiomatic characterizations of the solutions given for *DiG*, which will be the scope of a future paper.

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